EX 20 :

$U_0 = \int_0^1 \sqrt{1-x}.dx$

$U_n = \int_0^1 x^n.\sqrt{1-x}.dx$

1) $0 \le x \le 1 \Rightarrow -1 \le -x \le 0 \Rightarrow 0 \le 1-x \le 1$

$\Rightarrow 0 \le \sqrt{1-x} \le 1 \Rightarrow 0 \le x^n.\sqrt{1-x} \le x^n$

d'où $0 \le \int_0^1 x^n.\sqrt{1-x}.dx \le \int_0^1 x^n dx$

$\Rightarrow 0 \le U_n \le \left[\frac{x^{n+1}}{n+1}\right]_0^1 \Rightarrow 0 \le U_n \le \frac{1}{n+1}$

2) pour $n \ge 1$

on a : $0 \le U_n \le \frac{1}{n+1}$

$\lim_{n\to+\infty} \frac{1}{n+1} = 0$ d'où : $\lim_{n\to+\infty} U_n = 0$

3) a)

* $U_0 = \int_0^1 \sqrt{1-x}.dx = \left[-\frac{2}{3}(1-x)\sqrt{1-x}\right]_0^1$

$U_0 = \frac{2}{3}$

* $U_1 = \int_0^1 x.\sqrt{1-x}.dx$

intégrations par parties :

$\begin{cases} U'(x)=\sqrt{1-x} \xrightarrow{\int} U(x) = \frac{-2}{3}(1-x)\sqrt{1-x} \\ V(x) = x \xrightarrow{d} V'(x) = 1 \end{cases}$

$U_1 = \left[\frac{-2}{3}x(1-x)\sqrt{1-x}\right]_0^1 + \frac{2}{3}\int_0^1 (1-x)\sqrt{1-x}.dx$

$U_1 = \frac{2}{3}\int_0^1 (1-x)\sqrt{1-x}.dx = \frac{2}{3}\left(\int_0^1 \sqrt{1-x}.dx - \int_0^1 x\sqrt{1-x}.dx\right)$

$U_1 = \frac{2}{3}(U_0 - U_1)$

$\Rightarrow 3U_1 = 2U_0 - 2U_1 \Rightarrow 5U_1 = 2U_0 \Rightarrow U_1 = \frac{2}{5}U_0$

$U_1 = \frac{4}{15}$

b) pour $n \ge 2$

$U_n = \int_0^1 x^n \sqrt{1-x}.dx$

on pose : $\begin{cases} U'(x)=\sqrt{1-x} \longrightarrow U(x) = \frac{-2}{3}(1-x)\sqrt{1-x} \\ V(x) = x^n \longrightarrow V'(x) = n.x^{n-1} \end{cases}$

$U_n = \left[\frac{-2}{3}x^n(1-x)\sqrt{1-x}\right]_0^1 + \frac{2n}{3}\int_0^1 x^{n-1}(1-x)\sqrt{1-x}dx$

$U_n = \frac{2n}{3}\left(\int_0^1 (x^{n-1}\sqrt{1-x} - x^n\sqrt{1-x})dx\right)$

$U_n = \frac{2n}{3}\left(\int_0^1 (x^{n-1}\sqrt{1-x})dx - \int_0^1 (x^n\sqrt{1-x})dx\right)$

$U_n = \frac{2n}{3}(U_{n-1} - U_n)$

$\Rightarrow U_n = \frac{2n}{3}U_{n-1} - \frac{2n}{3}U_n$

$\Rightarrow 3U_n = 2nU_{n-1} - 2nU_n$

$\Rightarrow (3+2n).U_n = 2n.U_{n-1}$

pour n=1 : $(3+2)U_1 = 2n.U_0$

$5U_1 = 2U_0$ (vrai)

d'où : $(3+2n)U_n = 2nU_{n-1}$ $\forall n \in \mathbb{N}^*$

c) * $(3+4)U_2 = 4U_1 \Rightarrow U_2 = \frac{4}{7}U_1 \Rightarrow U_2 = \frac{16}{105}$

* $(3+6)U_3 = 6U_2 \Rightarrow U_3 = \frac{6}{9}U_2 \Rightarrow U_3 = \frac{2}{3}U_2 \Rightarrow U_3 = \frac{32}{315}$

EX 21 :

$n \ge 1$

$U_n = \int_0^1 \frac{x^n}{\sqrt{1+x^2}}dx$

1) $U_1 = \int_0^1 \frac{x}{\sqrt{1+x^2}}dx = \left[\sqrt{1+x^2}\right]_0^1 = \sqrt{2}-1$

2) pour $0 \le x \le 1$ on a : $1 \le \sqrt{1+x^2} \le 2$

$\Rightarrow 0 \le \frac{1}{\sqrt{1+x^2}} \le 1 \Rightarrow 0 \le \frac{x^n}{\sqrt{1+x^2}} \le x^n$

d'où $0 \le \int_0^1 \frac{x^n}{\sqrt{1+x^2}}dx \le \int_0^1 x^n dx$

$\Rightarrow 0 \le U_n \le \left[\frac{x^{n+1}}{n+1}\right]_0^1 \Rightarrow 0 \le U_n \le \frac{1}{n+1}$

Qarray.tn -Aqra Blech- Qarray.tn